Mechanics of Solids – Elastic-Plastic Deformations Notes

Learning Summary

1. Know the shapes of uniaxial stress-strain curves and the elastic-perfectly-plastic approximation (knowledge);
2. Know the kinematic and isotropic material behaviour models used to represent cyclic loading behaviour (knowledge);
3. Understand elastic-plastic bending of beams (comprehension) and be able to use equilibrium, compatibility and σ-ε behaviour to solve these types of problems for deformation and stress state (application);
4. Understand elastic-plastic torsion of shafts (comprehension) and be able to use equilibrium, compatibility and τ-γ behaviour to solve these types of problems for deformation and stress state (application);
5. Be able to determine residual deformations and residual stresses in beams under bending and shafts under torsion (application).

1. Introduction

When materials are subjected to an increasing load (or stress), the strain response is often such that there is a linear (elastic) region in the stress-strain plot followed by a non-linear (plastic) region, as shown schematically in Figure 1. The ability to predict this material behaviour is extremely important, within many applications, in order to determine maximum allowable loads, that can be applied to components. These allowable loads are usually based on both the displacement this load causes as well as the remaining (residual) deformation upon unloading.

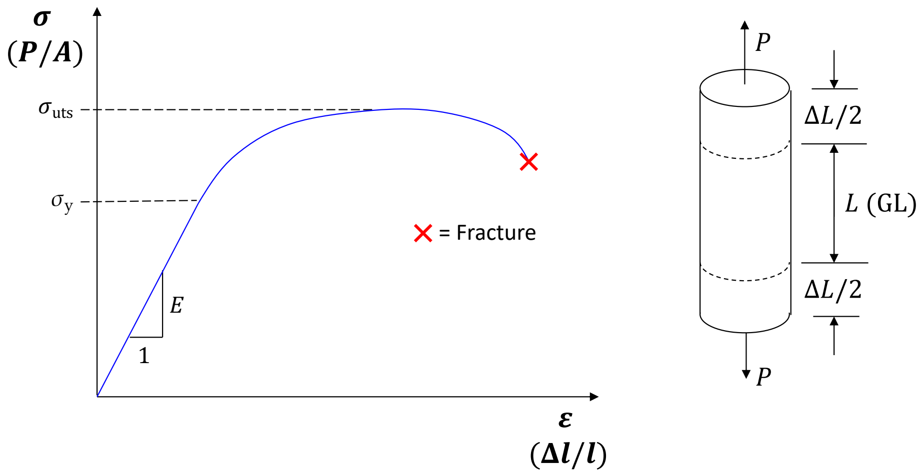


Figure 1

Several mathematical models can be used to estimate this material behaviour.

2. Elastic-Plastic Material Behaviour Models

Elastic-perfectly-plastic (EPP)

In this case, there is assumed to be no material hardening upon yield. I.e., once the yield stress, σ_y, is reached, further straining causes no further increase in stress, as shown in Figure 2.

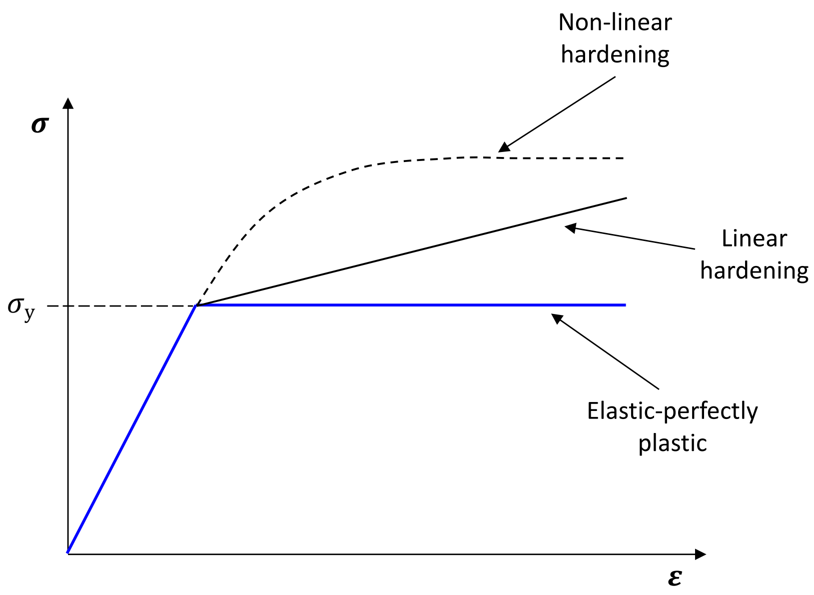


Figure 2

Figure 2 shows an EPP stress-strain curve for a material in tension. However, this behaviour is also applicable in compression. I.e., if the loading is reversed, the behaviour shown in Figure 2 can be extended to that shown in Figure 3, where it can be seen that the stress magnitude increases in compression until the compressive yield stress, $-\sigma_y$, is reached, after which no further change to the stress response occurs with increasing compressive strain magnitude.

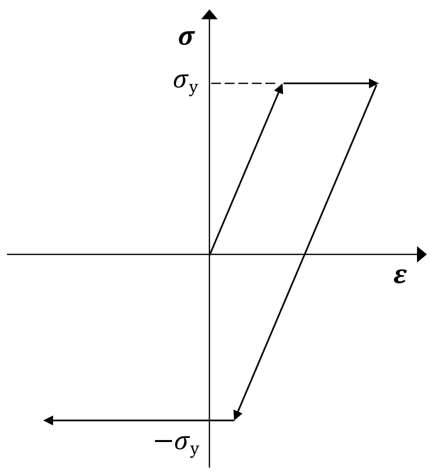


Figure 3

If the loading is then cycled between tension and compression, the material will continue to behave in the same way (regardless of any previous plastic deformation) resulting in the hysteresis loop shown in Figure 4.

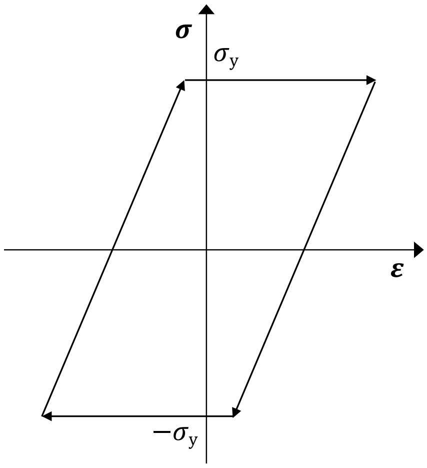


Figure 4

For EPP material behaviour, as loading conditions cause yielding (plasticity), there is no change to the yield surface, shown in Figure 5 (in blue for the von Mises yield criterion and in red for the Tresca yield criterion), in the principal stress-space.

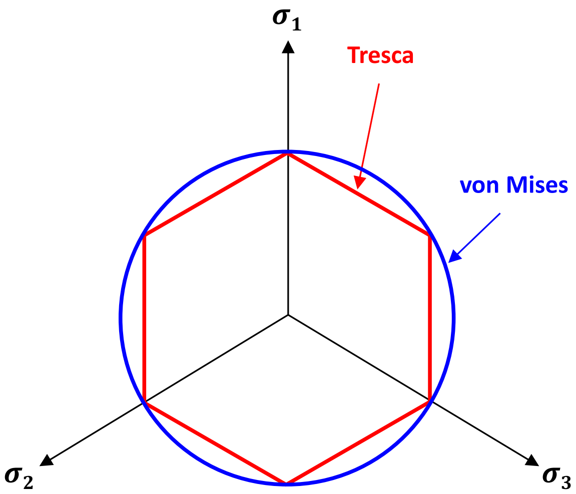


Figure 5

EPP, is a good material model for mild steel, for example, which demonstrates moderate plasticity.

Isotropic Hardening

For materials which harden, as shown in tension, for both linear and non-linear cases, in Figure 2, this hardening behaviour can also be observed as changes to the yield surface. For the case of isotropic hardening behaviour, when the loading state, shown by the red arrow in Figure 6a, reaches the point of causing yielding (plasticity), i.e., point a, the yield surface will begin to grow. Point a, the yield point, is also shown on the equivalent stress-strain curve in Figure 6c. As the loading state is further increased to point b, as shown in Figure 6b and 6c, the yield surface remains centred at the same position, but its radius grows in all directions by an amount governed by the magnitude of the loading.

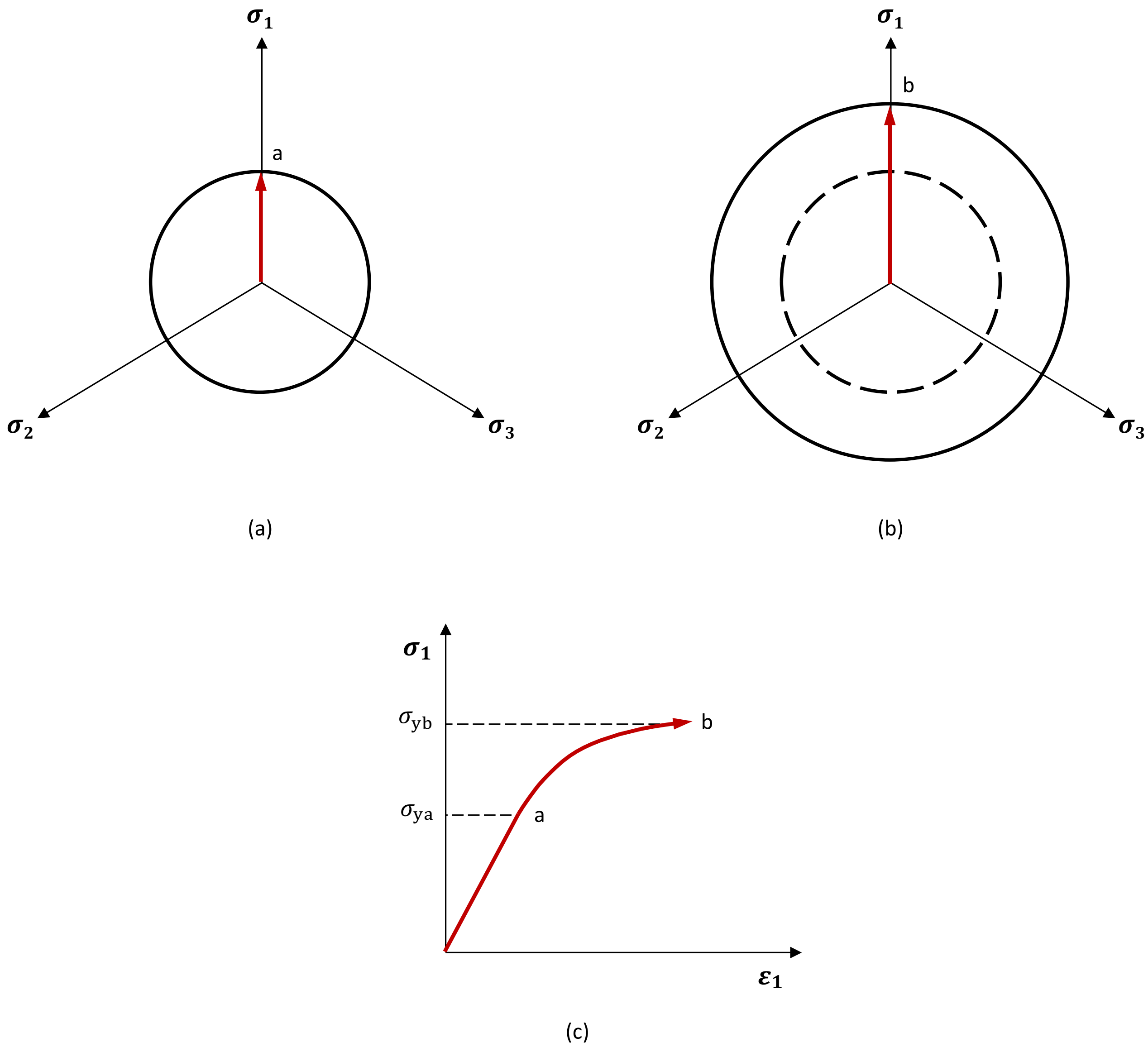


Figure 6

If the loading is then reversed, as shown in Figure 7a, further plasticity (and therefore hardening) does not occur until the magnitude of the reserved loading is such that the edge of the increased yield surface, point c, is reached. This can also be represented on the equivalent stress-strain curve, as shown in Figure 7c. This position of compressive yield is at a larger load magnitude than if loaded in this direction originally. This is due to the growth of the yield surface (isotropic hardening) during the prior tensile loading. As the compressive load magnitude is further increased to point d, as shown in Figure 7b and 7c, the yield surface again remains centred at the same position, but its radius grows further in all directions by an amount governed by the magnitude of the loading.

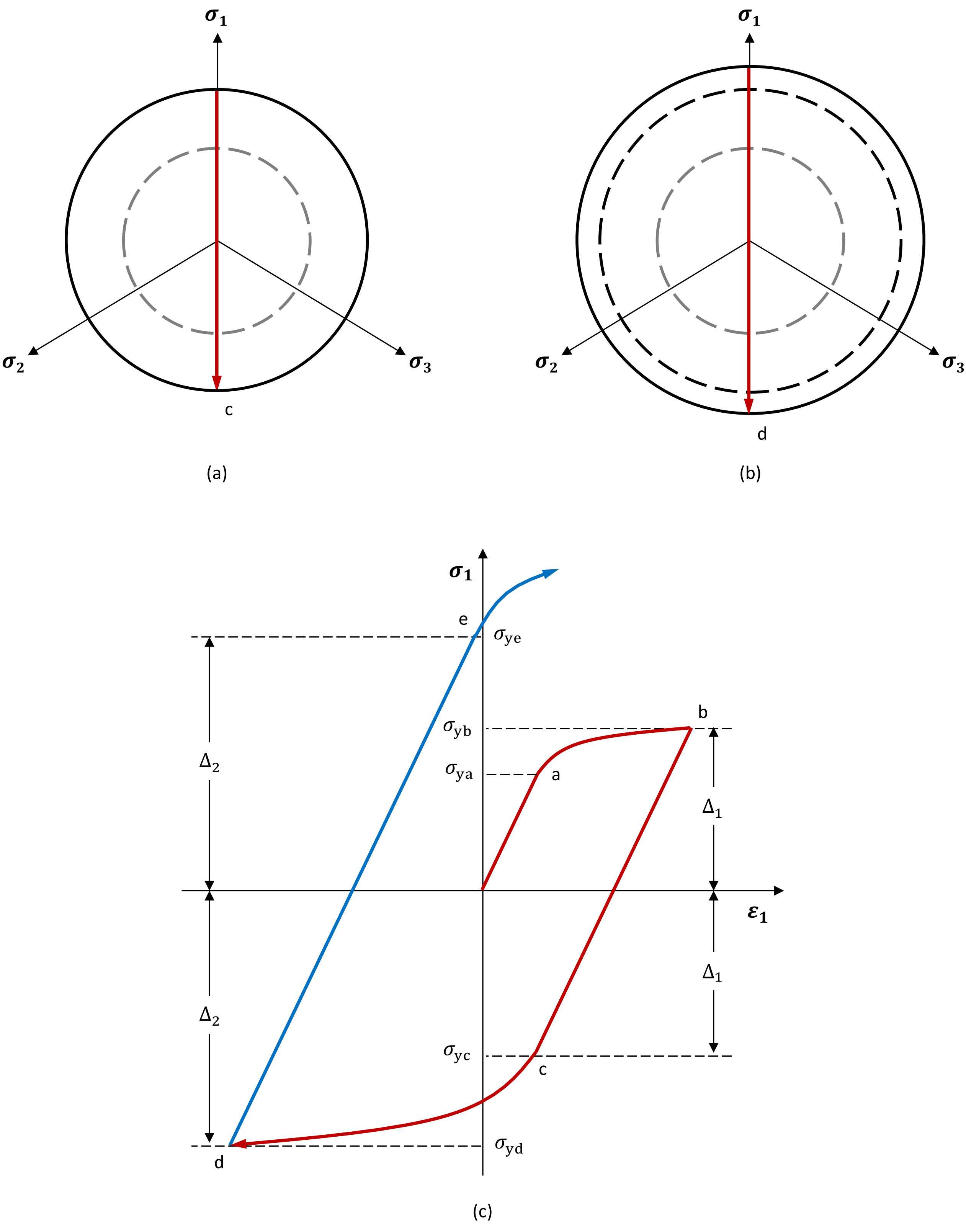


Figure 7

The blue loading curve shown in Figure 7c shows how the material would behave under a further tensile loading and shows that again, further plasticity (and therefore hardening) does not occur until the magnitude of the loading is such that the edge of the increased yield surface, point e, is reached.

Kinematic Hardening

In the case of kinematic hardening behaviour, when the loading state reaches the point of causing yielding within the material, i.e., point a in Figure 8a, the yield surface begins moves in the direction of the loading. Point a, the yield point, is also shown on the equivalent stress-strain curve in Figure 8c. As the load is further increased to point b, shown in Figure 8b and 8c, the yield surface remains the same size (diameter of $2\sigma_{\mathrm{ya}}$) but moves in the direction of the loading by an amount governed by the magnitude of the loading.

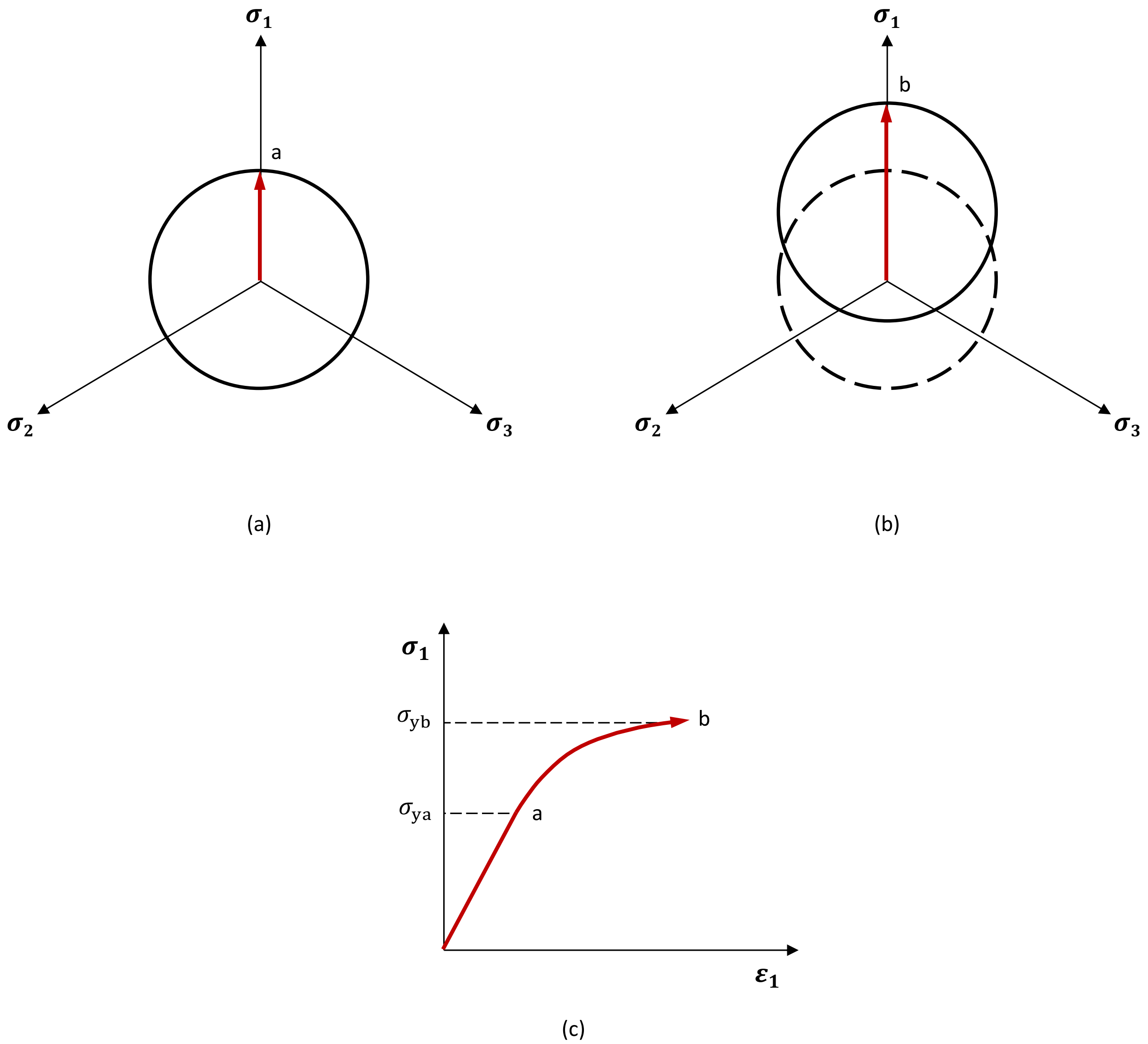


Figure 8

If the loading is then reversed, as shown in Figure 9a, further plasticity (and therefore hardening) will occur at position c. This can also be represented on the equivalent stress-strain curve, as shown in Figure 9c. This position of compressive yield is now at a lower load magnitude than if loaded in this direction originally, due to the movement of the yield surface (kinematic hardening) during the prior tensile loading. As the compressive load magnitude is further increased to point d, as shown in Figure 9b and 9c, the yield surface remains the same size, but its centre again moves in the direction of the loading by an amount governed by the magnitude of the loading.

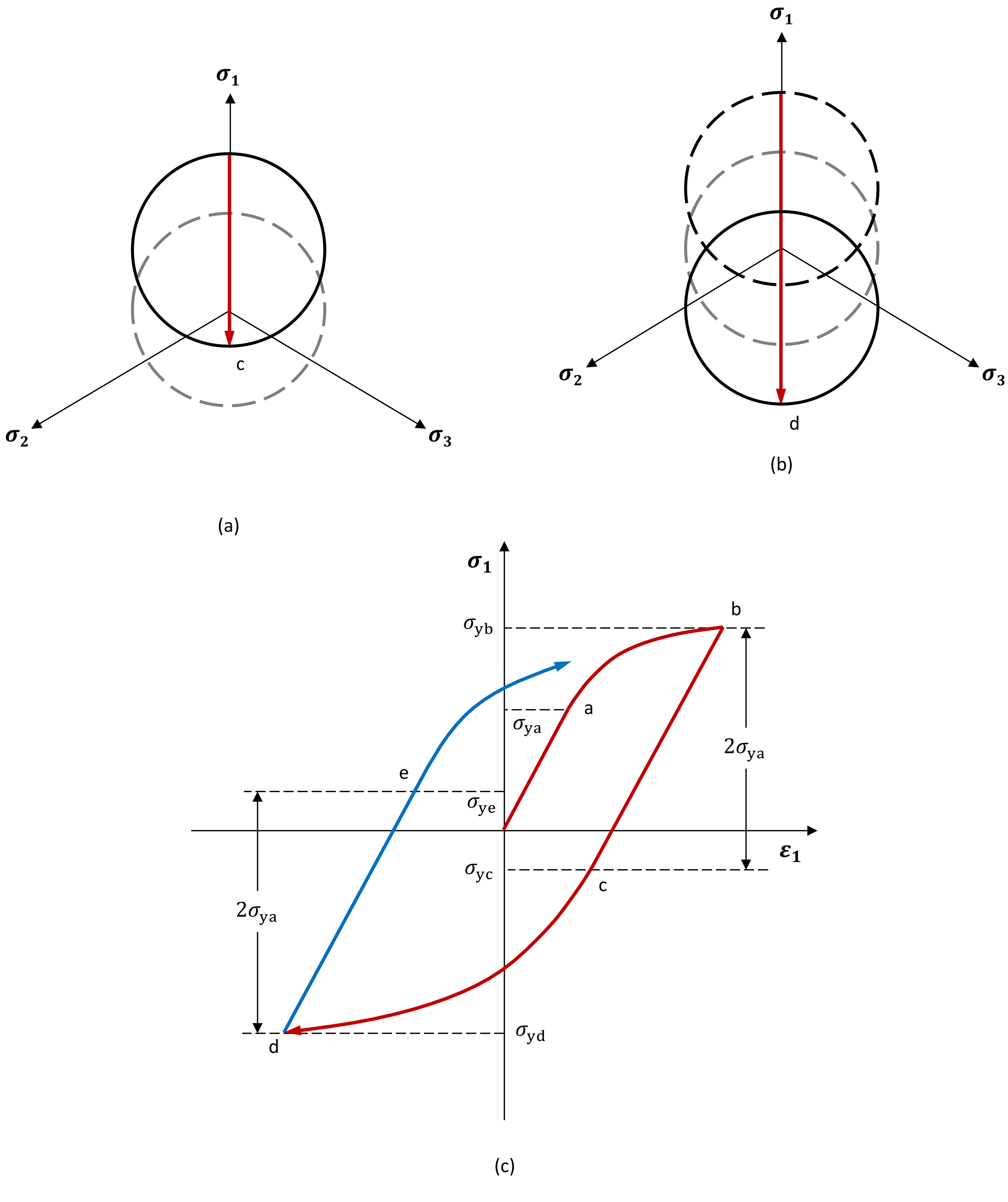


Figure 9

The blue loading curve shown in Figure 9c shows how the material would behave under a further tensile loading and shows that again, further plasticity (and therefore hardening) occurs at a lower stress magnitude, point e, than in the previous tensile and compressive loadings, due to the movement of the yield surface (kinematic hardening) during the prior compressive loading.

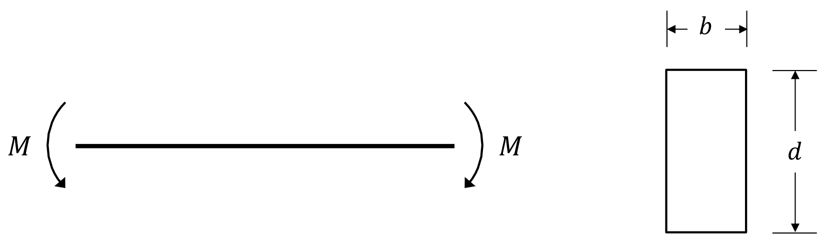
In the descriptions of isotropic and kinematic hardening above, the loading was chosen to be in the 1-direction, however, it is important to note that the applied loading could be in any direction, with the concepts described above remaining valid. It should also be noted that in both cases, the von Mises yield criterion and non-linear hardening were chosen for demonstrative purposes, but the same concepts would apply in the case of the Tresca yield criterion and/or linear hardening, respectively.

In reality, it is not common for materials to harden in a purely isotropic or kinematic manner, but rather a mixture of these. There are material behaviour models (e.g., the unified visco-plasticity model) which account for both isotropic and kinematic hardening. Also, whereas isotropic and kinematic hardening represent growth and movement of the yield surface, respectively, other material hardening models represent a change in the shape of the yield surface.

In the following analyses related to the elastic-plastic deformation of components (e.g., beams in bending and torsion of shafts), only the EPP material behaviour model will be considered.

3. Elastic-Plastic Bending of Beams

Figure 10a shows a beam which is subjected to a bending moment, M. The rectangular cross-sectional area of the beam is shown in Figure 10b.



(a) (b)

Figure 10

Assuming that the magnitude of the bending moment is not high enough to cause plasticity (yielding) within the beam, the elastic beam bending equation can be used to describe the stress distribution, as a function of y (distance from the neutral axis), as:

$$\sigma = \frac{My}{I} \tag{1}$$

In this elastic case then, the stress distribution, as a function of y, throughout the cross-section, is linear, as shown in Figure 11.

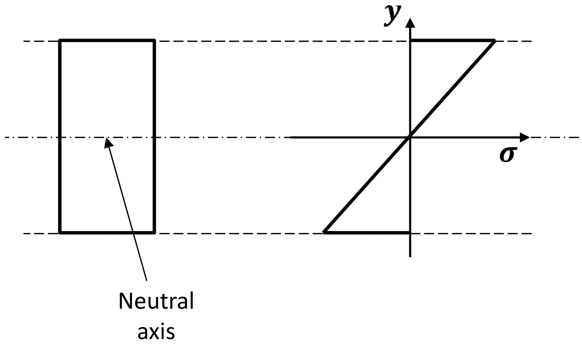


Figure 11

If the bending moment is increased to a magnitude which is just high enough to induce plasticity within the beam, this plasticity will occur at the positions furthest away from the neutral axis, i.e., at the positions of maximum y magnitude (top and bottom edges of the cross-section). As the bending moment is further increased, the plasticity spreads from the outer edges, to further within the cross-section (towards the neutral axis) as shown in Figure 12. As can be seen from Figure 12, the material behaviour demonstrated is elastic perfectly-plastic, as once the material has yielded, at $y > a$ and $y < -a$, no further increase in stress magnitude is observed.

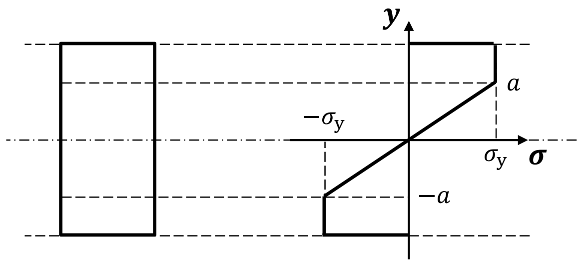


Figure 12

Moment equilibrium can be used to relate the applied bending moment, M, to the position as which yielding occurs, a, as:

$$M = \int_A y\sigma \mathrm{d}A \tag{2}$$

Equation (2) shows that the sum of the moments caused as a result of the stress, σ, in each unit of area, $\mathrm{d}A$, in the cross-section, must be equal to the applied bending moment, M. This can be seen in the right-hand side of the above equation as stress, σ, multiplied by area, A, gives force, F, and force multiplied by perpendicular distance, y, gives bending moment. I.e., for each unit of area, $\mathrm{d}A$:

$$\sigma \times \mathrm{d}A = \mathrm{dF}$$

and

$$\mathrm{d}F \times y = \mathrm{d}M$$

Equation (2) can be rewritten as:

$$M = \int_{-d/2}^{d/2} y\sigma b\mathrm{d}y \tag{3}$$

where b and d are the width (or breadth) and depth of the cross-section, respectively, as shown in Figure 10, and where:

$$\mathrm{d}A = b\mathrm{d}y$$

Recognising the symmetry about the neutral axis in the stress distribution magnitudes, equation (3) can be rewritten as:

$$M = 2\int_{0}^{d/2} y\sigma b\mathrm{d}y$$

Substituting the expressions for stress for each of the elastic ($0 > y > a$) and plastic ($a > y > d/2$) regions, i.e., $\sigma_\mathrm{y}\frac{y}{a}$ and σ_y, respectively, into this gives:

$$M = 2\int_{0}^{a} y\left(\sigma_\mathrm{y}\frac{y}{a}\right) b\mathrm{d}y + \int_{a}^{d/2} y\sigma_\mathrm{y} b\mathrm{d}y$$

$$= 2b\sigma_\mathrm{y}\left(\frac{d^2}{8} - \frac{a^2}{6}\right)$$

In order for the radius of curvature of the beam, R, due to the applied bending moment, M, to be calculated, both compatibility and a stress-strain relationship are required. As the region of the cross-section between $-a < y < a$ has only behaved elastically, the elastic beam bending equation can be applied. I.e.:

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\therefore \frac{y}{R} = \frac{\sigma}{E} = \varepsilon$$

$$\therefore R = \frac{y}{\varepsilon} \tag{4}$$

where ε is the strain related to the stress σ.

As the beam behaves as one body, the entirety of the beam (both the elastic and plastic regions) must share this common radius of curvature, R. This is the **compatibility requirement** mentioned above.

Again, as the region of the cross-section between $-a < y < a$ has only behaved elastically, Hooke's law applies here, and so:

$$\sigma = E\varepsilon$$

This is the required **stress-strain** relationship mentioned above.

Rearranging for ε and substituting this into equation (4):

$$R = \frac{Ey}{\sigma} \tag{5}$$

Substituting values for y and σ, from within the elastic region, into this equation, allows for R to be calculated. A convenient value of y to use is a (which is the outermost point of the elastic region), for which the corresponding value of σ is σ_{y}. Therefore:

$$R = \frac{Ea}{\sigma_y}$$

As plasticity has occurred within the beam during loading, on unloading the stress distribution and radius of curvature will not return to zero. Rather a residual stress distribution and residual radius of curvature will remain. If we assume that the stress change which occurs on unloading is purely elastic, then the stress change, $\Delta\sigma$, can be calculated from equation (1) as:

$$\Delta\sigma = \frac{\Delta M y}{I}$$

The maximum stress change, $\Delta\sigma_{\mathrm{max}}$, will therefore occur at y_{max}, and so:

$$\Delta\sigma_{\mathrm{max}} = \frac{\Delta M \times y_{\mathrm{max}}}{I} = \frac{-M \times \pm {}^{d}/_{2}}{I}$$

where the change in bending moment on unloading, ΔM, is $-M$ and $y_{\mathrm{max}} = \pm {}^{d}/_{2}$.

Therefore, at $y = {}^{d}/_{2}$ (top edge):

$$\Delta\sigma_{(y={}^{d}/_{2})} = \frac{-Md}{2I}$$

and at $y = -{}^{d}/_{2}$ (bottom edge):

$$\Delta\sigma_{(y=-{}^{d}/_{2})} = \frac{Md}{2I}$$

As the unloading behaviour has been assumed to be elastic, the stress variation between these two values, relating to the top and bottom edges, will be linear, as shown by the unloading part of Figure 13.

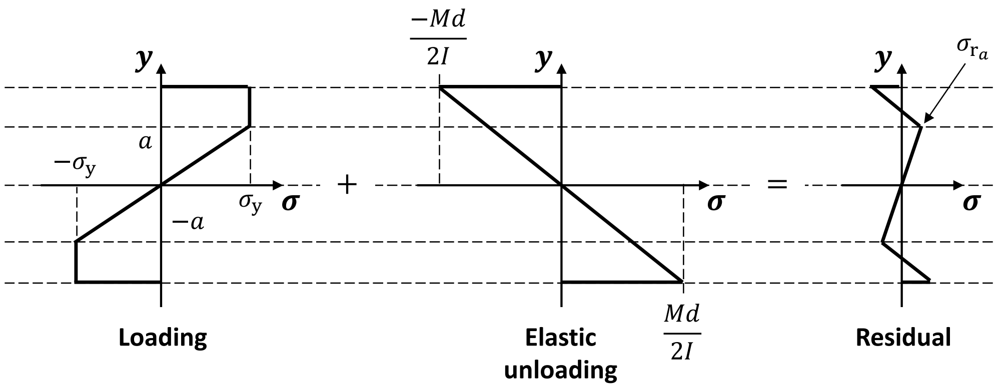


Figure 13

Figure 13 also shows that by summing the loaded stress distribution on the cross-section with the stress change which occurs on unloading, the residual stress distribution can be obtained.

It is clear that the residual stresses are well below the yield stress, so reverse yielding does not occur, and therefore the elastic unloading assumption made, is correct.

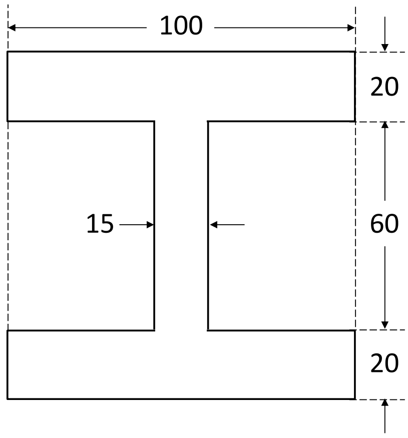
This residual stress distribution will be accompanied by a residual radius of curvature, which can be calculated by substituting unloaded beam values for y and σ into equation (5), which again relate to a position that has only been subjected to elastic behaviour. As before (under loaded conditions), a convenient value of y to use is a (which is the outermost point of the elastic region), for which the corresponding value of σ can be taken from the residual stress distribution given in Figure 13 and is shown labelled as σ_{r_a} (i.e., the residual stress, σ_r, at position a).

On releasing the moment, the radius of curvature increases. This change of curvature is called 'spring back' and is particularly important when bending beams to specified radii of curvature.

4. Worked Example – Elastic-Perfectly Plastic I-Beam Subjected to a Pure Bending Moment

Problem

Figure 14 shows the cross-section of a straight I-beam which is subjected to a pure bending moment, M.



All dimensions in mm

Figure 14

Calculate:

a. the maximum allowable value of M if the web of the section is not to be subjected to any plasticity.
b. the stress distribution and radius of curvature upon the application of M
c. the stress distribution and radius of curvature upon unloading

The material can be assumed to be elastic-perfectly-plastic with a yield stress, $\sigma_y = 215$ MPa, and Young's Modulus, $E = 200$ GPa.

Solution

As it is known that the full depths of each flange are allowed to yield, but the web is to remain fully elastic, the resulting stress distribution due to the application of the maximum allowable bending moment, M, is as shown in Figure 15.

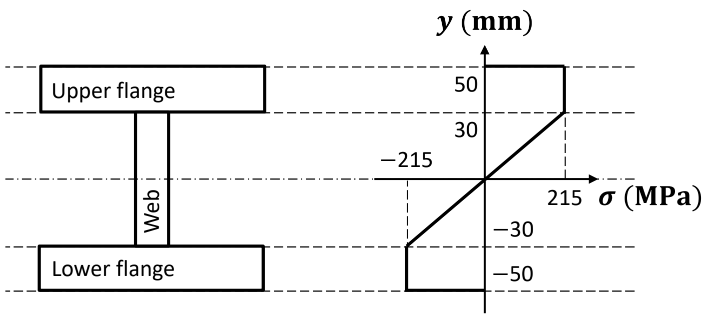


Figure 15

Moment Equilibrium

Balancing the moments due to stresses in the elastic and plastic regions with the applied moment:

$$M = \int_A y\sigma dA$$

$$= \int_{-d/2}^{d/2} y\sigma b\mathrm{d}y$$

Due to the symmetry of the stress distribution magnitude about the neutral axis and substituting in the elastic and plastic terms for σ, this can be rewritten as:

$$M = 2\left(\int_0^a y\left(\sigma_\mathrm{y}\frac{y}{a}\right) b_w \mathrm{d}y + \int_a^{d/2} y\sigma_\mathrm{y} b_f \mathrm{d}y\right)$$

$$= 2\sigma_\mathrm{y}\left(\frac{b_w}{a}\int_0^a y^2\mathrm{d}y + b_f\int_a^{d/2} y\mathrm{d}y\right)$$

where b_w and b_f are the widths of the web and flange sections of the beam, respectively, and a is the value of y where the cross-section transitions from the web to the flange.

Therefore,

$$M = 2\sigma_\mathrm{y}\left(\frac{b_w}{a}\left[\frac{y^3}{3}\right]_0^a + b_f\left[\frac{y^2}{2}\right]_a^{d/2}\right)$$

$$= 2\sigma_\mathrm{y}\left(b_w\frac{a^2}{3} + b_f\left(\frac{d^2}{8} - \frac{a^2}{2}\right)\right)$$

$$\therefore \boldsymbol{M = 36,335,000\ \mathbf{Nmm} = 36.34\ \mathbf{kNm}}$$

Compatibility

As the region of the cross-section between $-a < y < a$ has only behaved elastically, the elastic beam bending equation can be applied and rearranged to give:

$$R = \frac{y}{\varepsilon}$$

As the beam behaves as one body, this expression for R can be applied to any value of y.

Stress-Strain Relationship

Again, as the region of the cross-section between $-a < y < a$ has only behaved elastically, Hooke's law applies here, which can be substituted into the above expression for R to give:

$$R = \frac{Ey}{\sigma}$$

Substituting values for y and σ from the outermost point of the elastic region gives:

$$R_{\text{load}} = \frac{Ea}{\sigma_y}$$

$$\therefore \boldsymbol{R_{\text{load}} = 27,906.98\ \text{mm} = 27.91\ \text{m}}$$

Unloading

Assuming that the stress change caused by unloading is purely elastic, then from the elastic beam bending equation:

$$\Delta\sigma = \frac{\Delta M \times y}{I}$$

Therefore, at the top and bottom edges:

$$\Delta\sigma_{(y={}^{d}/_{2})} = \frac{-M \times {}^{d}/_{2}}{I} = \frac{-Md}{2I} = \frac{-36{,}335{,}000 \times 100}{2 \times 6{,}803{,}333.33} = -267.04\ \text{MPa}$$

and

$$\Delta\sigma_{(y=-{}^{d}/_{2})} = \frac{-M \times -{}^{d}/_{2}}{I} = \frac{Md}{2I} = 267.04\ \text{MPa}$$

Where:

$$I = \left(\frac{bd^3}{12}\right)_{outer} - \left(\frac{bd^3}{12}\right)_{gaps} = \frac{100 \times 100^3}{12} - 2\left(\frac{42.5 \times 60^3}{12}\right) = 8{,}333{,}333.33\ \text{mm}^4 - 1{,}530{,}000\ \text{mm}^4$$

$$= 6{,}803{,}333.33\ \text{mm}^4$$

Since the unloading behaviour has been assumed to be elastic, the stress variation between these two values, relating to the top and bottom edges, will be linear, as shown in the unloading section of Figure 16.

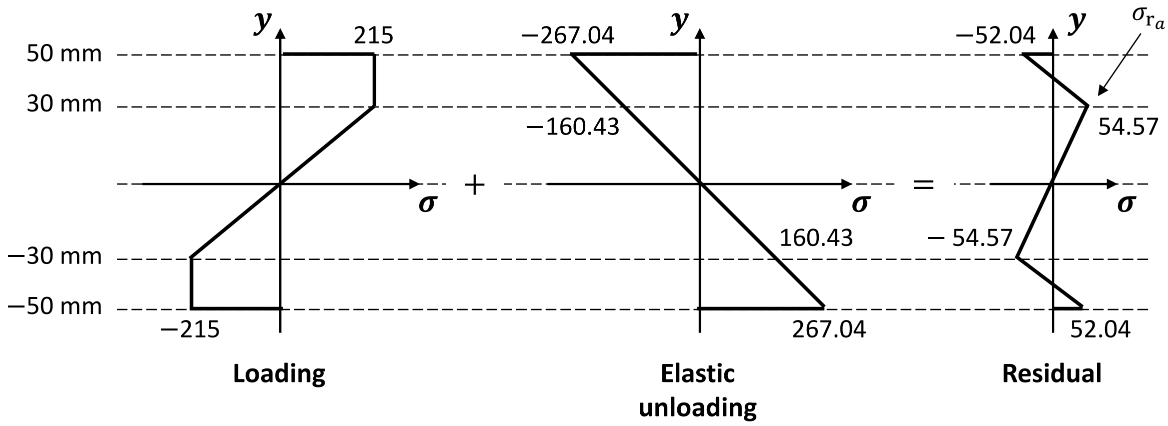


Figure 16

The equation of the linear unloading relationship is:

$$y = m\sigma + c \tag{6}$$

where, as the line passes through the origin:

$$c = 0$$

and the gradient, m, can be calculated by choosing a corresponding set of y and σ values, e.g., those applicable to the top edge, i.e., $y = 50$ mm and $\sigma = -267.04$ MPa and substituting these into equation (6) as:

$$\therefore 50 = m \times -267.04$$

$$\therefore m = -0.187$$

This value for m allows for the interpolation of the unloading stress distribution to determine the σ value at $y = 30$ mm, from equation (6) as:

$$30 = -0.187\sigma$$

$$\therefore \sigma = -160.43$$

The loading and unloading stress distributions can now be summed together in order to determine the residual stress distribution as follows.

At $y = 50$ mm:

$$\sigma_{r_{50}} = \sigma_{\text{load}_{50}} + \sigma_{\text{unload}_{50}} = 215 \text{ MPa} - 267.04 \text{ MPa} = -52.04 \text{ MPa}$$

At $y = 30$ mm:

$$\sigma_{r_{30}} = \sigma_{\text{load}_{30}} + \sigma_{\text{unload}_{30}} = 215 \text{ MPa} - 160.43 \text{ MPa} = 54.57 \text{ MPa}$$

At $y = 0$ mm:

$$\sigma_{r_0} = \sigma_{\text{load}_0} + \sigma_{\text{unload}_0} = 0 \text{ MPa} - 0 \text{ MPa} = 0 \text{ MPa}$$

At $y = -30$ mm:

$$\sigma_{r_{-30}} = \sigma_{\text{load}_{-30}} + \sigma_{\text{unload}_{-30}} = -215 \text{ MPa} + 160.43 \text{ MPa} = -54.57 \text{ MPa}$$

At $y = -50$ mm:

$$\sigma_{r_{-50}} = \sigma_{\text{load}_{-50}} + \sigma_{\text{unload}_{-50}} = -215 \text{ MPa} + 267.04 \text{ MPa} = 52.04 \text{ MPa}$$

The residual section of Figure 16 shows this in graphical form.

As the residual stresses are well below the yield stress (±215 MPa), reverse yielding will not occur, and therefore the elastic unloading assumption made, is correct.

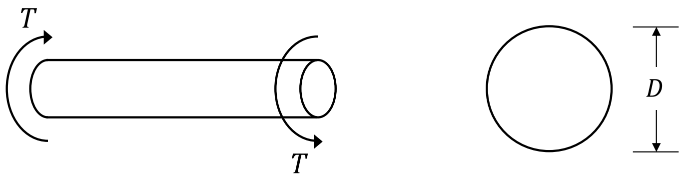
This residual stress distribution will be accompanied by a residual radius of curvature, which can be calculated by substituting unloaded beam values for y and σ into the same equation as above for the loaded equivalent, which again relate to a position that has only been subjected to elastic behaviour. As before (under loaded conditions), a convenient value of y to use is $a = 30$ mm (which is the outermost point of the elastic region), for which the corresponding value of σ can be seen, from Figure 16, to be 54.57 MPa. I.e.:

$$R_{\text{unload}} = \frac{Ea}{\sigma_{\text{r}_a}}$$

$$\therefore \boldsymbol{R}_{\textbf{unload}} = \mathbf{109{,}950.52\ mm = 109.95\ m}$$

5. Elastic-Plastic Torsion of Shafts

Figure 17a shows a shaft which is subjected to a torque, T. The circular cross-sectional area of the shaft is shown in Figure 17b.



(a) (b)

Figure 17

Assuming that the magnitude of the torque is not high enough to cause plasticity (yielding) within the shaft, the elastic shaft torsion equation can be used to describe the shear stress distribution, as a function of r (radius of the shaft), as:

$$\tau = \frac{Tr}{J} \tag{7}$$

In this elastic case then, the shear stress distribution, as a function of r, throughout the cross-section is linear, as shown in Figure 18.

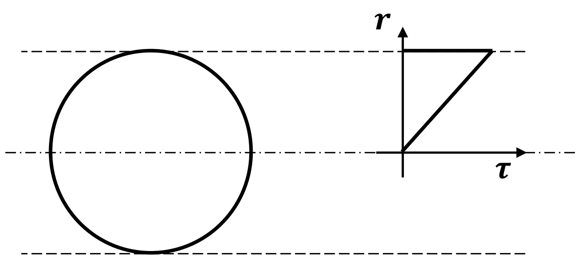


Figure 18

If the torque is increased to a magnitude which is just high enough to induce plasticity in the shaft, this plasticity will occur at the positions furthest away from the centre of the cross-section, i.e., at the positions of maximum r magnitude (circumference of the cross-section). As the torque is further increased, the plasticity spreads from the outer edge, to further within the cross-section (towards the centre) as shown in Figure 19. As can be seen from Figure 19, the material behaviour demonstrated is elastic perfectly-plastic, as once the material has yielded, at $r > a$, no further increase in stress magnitude is observed.

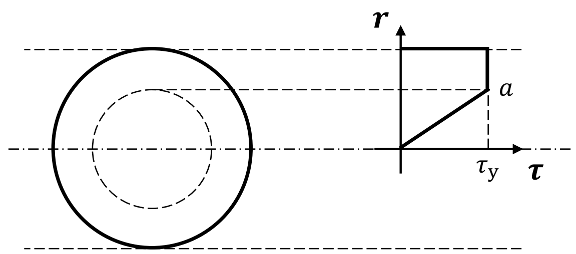


Figure 19

Torque equilibrium can be used to relate the applied torsion, T, to the position as which yielding occurs, a, as:

$$T = \int_A r\tau dA \tag{8}$$

Equation (8) shows that the sum of the torques caused as a result of the shear stress, τ, in each unit of area, $\mathrm{d}A$, in the cross-section, must be equal to the applied torque, T. This can be seen in the right-hand side of the above equation as shear stress, τ, multiplied by area, A, gives units of force, and force multiplied by perpendicular distance, r, gives torque.

Equation (8) can be rewritten as:

$$T = 2\pi \int_0^R \tau r^2 \mathrm{d}r \tag{9}$$

where R is the outer radius of the cross-section, and:

$$\mathrm{d}A = 2\pi r \mathrm{d}r$$

Substituting the expressions for shear stress for each of the elastic ($0 > r > a$) and plastic ($a > r > R$) regions, i.e., $\tau_y \frac{r}{a}$ and τ_y, respectively, into equation (9) gives:

$$T = 2\pi \int_0^a \left(\tau_y \frac{r}{a}\right) r^2 \mathrm{d}r + 2\pi \int_a^R \tau_y r^2 \mathrm{d}r$$

$$= 2\pi\tau_y \left(\frac{R^3}{3} - \frac{a^3}{12}\right)$$

In order for the twist, θ, of the shaft, due to the applied torque, T, to be calculated, both compatibility and a shear stress-shear strain relationship are required. As the region of the cross-section between $0 < r < a$ has only behaved elastically, the elastic shaft torsion equation can be applied. I.e.:

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

$$\therefore \frac{r\theta}{L} = \gamma \left(= \frac{\tau}{G}\right)$$

$$\therefore \theta = \frac{\gamma L}{r} \tag{10}$$

where γ is the shear strain related to the shear stress τ.

As the shaft behaves as one body, the entirety of the shaft (both the elastic and plastic regions) must share this common twist, θ. This is the **compatibility requirement** mentioned above.

Again, as the region of the cross-section between $0 < r < a$ has only behaved elastically, Hooke's law applies here, therefore:

$$\tau = G\gamma$$

This is the required **shear stress-shear strain** relationship mentioned above. Rearranging for γ and substituting this into equation (10):

$$\theta = \frac{\tau L}{Gr} \tag{11}$$

Substituting values for r and τ, from within the elastic region, into this equation, allows for θ to be calculated. A convenient value of r to use is a (which is the outermost point of the elastic region), for which the corresponding value of τ is τ_{y}. Therefore:

$$\theta = \frac{\tau_y L}{Ga}$$

As plasticity has occurred within the shaft during loading, on unloading the shear stress distribution and twist will not return to zero. Rather a residual shear stress distribution and residual twist will remain. If we assume that the shear stress change which occurs on unloading is purely elastic, then the shear stress change, $\Delta\tau$, can be calculated from equation (7) as:

$$\Delta\tau = \frac{\Delta T \times r}{J}$$

The maximum shear stress change, $\Delta\tau_{\mathrm{max}}$, will therefore occur at r_{max}, and so:

$$\Delta\tau_{\mathrm{max}} = \frac{\Delta T \times r_{\mathrm{max}}}{J} = \frac{-TR}{J}$$

where the change in torque on unloading, ΔT, is $-T$ and $r_{\mathrm{max}} = R$.

As the unloading behaviour has been assumed to be elastic, the shear stress variation through the cross-section will be linear, as shown by the unloading part of Figure 20.

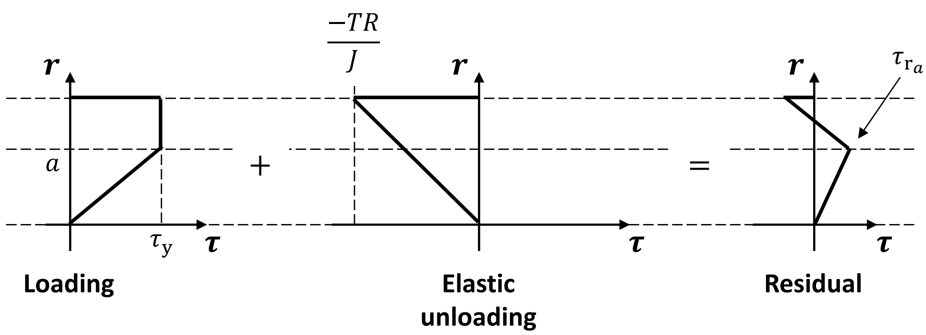


Figure 20

Figure 20 also shows that by summing the loaded shear stress distribution on the cross-section with the shear stress change which occurs on unloading, the residual shear stress distribution can be obtained.

It is clear that the residual shear stresses are well below the yield shear stress, so reverse yielding does not occur, and therefore the elastic unloading assumption made, is correct.

This residual shear stress distribution will be accompanied by a residual shaft twist, which can be calculated by substituting unloaded beam values for r and τ into equation (11), which again relate to a position that has only been subjected to elastic behaviour. As before (under loaded conditions), a convenient value of r to use is a (which is the outermost point of the elastic region), for which the corresponding value of τ can be taken from the residual shear stress distribution given in Figure 20 and is shown labelled as τ_{r_a} (i.e., the residual shear stress, τ_{r}, at position a).